NOTE / NOTE

A comparison of alternative methods for estimating the self-thinning boundary line

Lianjun Zhang, Huiquan Bi, Jeffrey H. Gove, and Linda S. Heath

Abstract: The fundamental validity of the self-thinning "law" has been debated over the last three decades. A long-standing concern centers on how to objectively select data points for fitting the self-thinning line and the most appropriate regression method for estimating the two coefficients. Using data from an even-aged *Pinus strobus* L. stand as an example, we show that quantile regression (QR), deterministic frontier function (DFF), and stochastic frontier function (SFF) methods have the potential to produce an upper limiting boundary line above all plots for the maximum size–density relationship, without subjectively selecting a subset of data points based on predefined criteria. On the other hand, ordinary least squares (OLS), corrected ordinary least squares (COLS), and reduced major axis (RMA) methods are sensitive to the data selected for model fitting and may produce self-thinning lines with inappropriate slopes. However, statistical inference is very difficult with the DFF and QR methods. Although SFF produces a self-thinning line lower than the upper limiting boundary line because of the nature of the method, it can easily produce the statistics for inference on the model coefficients, given that there are no significant departures from underlying distributional assumptions.

Résumé : La validité fondamentale de la « loi » d'autoéclaircie a été débattue au cours des trois dernières décennies. Une préoccupation de longue date porte sur la façon de sélectionner objectivement les données pour ajuster la droite d'autoéclaircie et sur la méthode de régression la plus appropriée pour estimer les deux coefficients. À l'aide de données provenant d'un peuplement équienne de *Pinus strobus* L. comme exemple, les auteurs montrent que les méthodes de régression quantile (RQ), de fonction déterministe frontière (FDF) et de fonction stochastique frontière (FSF) ont la capacité de produire une droite qui constitue la limite supérieure au-dessus de toutes les parcelles pour la relation maximale de la dimension en fonction de la densité sans sélectionner subjectivement un sous-ensemble de points basés sur des critères prédéfinis. D'un autre côté, la méthode ordinaire des moindres carrés, la méthode ordinaire corrigée des moindres carrés et la méthode des axes majeurs réduits sont sensibles aux données sélectionnées pour l'ajustement du modèle et peuvent produire des droites d'autoéclaircie avec des pentes inappropriées. Cependant, l'inférence statistique est très difficile avec les méthodes FDF et RQ. Même si la méthode FSF produit une droite d'autoéclaircie plus basse que la droite qui constitue la limite supérieure maximale, à cause de la nature de la méthode, elle peut facilement produire les statistiques pour déduire les coefficients du modèle, étant donné qu'il n'y a pas de démarcation significative des hypothèses sous-jacentes de distribution.

[Traduit par la Rédaction]

Introduction

Self-thinning is a dynamic equilibrium between plant growth and death at crowding density and is governed by the so-called "self-thinning rule" or "–3/2 power law" (Yoda et al. 1963; Westoby 1984). The rule states that, in logarithmic scales, the relationship between average plant size and stand density is a straight line (i.e., self-thinning line or maximum size–density relationship) for a stand undergoing density-related mortality. Historically, this self-thinning line has been expressed by relating either mean plant biomass or total stand biomass to stand density on logarithmic scales with a constant slope of –1.5 (for mean biomass) or –0.5 (for total biomass) in plant population ecology (Yoda et al. 1963; Westoby 1984). Stem volume has been used often in lieu of biomass for tree species. A long tradition in forestry has been to relate mean diameter to stand density, that is, $\log N = a - 1.6\log D$, where N is number of trees per unit area, D is

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quadratic mean diameter, and a is a constant (Reineke 1933). A recent variation of this expression is $\log D = b - 0.6\log N$ (Jack and Long 1996), where b is a constant. The intercept of the self-thinning line varies with species, but only within narrow logarithmic limits (White 1985), while the slope parameter is apparently consistent regardless of species, age, and site quality (Jack and Long 1996). Consequently, the self-thinning rule has been considered one of the most important principles in plant population ecology (Drew and Flewelling 1977; Long and Smith 1984; Jack and Long 1996).

Since the late 1980s the controversy about the fundamental validity of the self-thinning rule has been more intense. The debate has primarily focused on whether the slope of the self-thinning line is invariant (Zeide 1987; Weller 1989; Lonsdale 1990; Osawa and Allen 1993; Guo and Rundel 1998). A long-standing concern centers on the most appropriate methods for data selection and parameter estimation in the self-thinning equations (Weller 1989; Bi and Turvey 1997; Bi et al. 2000). It is emphasized that data used to estimate the maximum size–density relationship should be carefully selected. Incorrectly including data points from stands of mean density (in number of plants per area) that have not yet begun to thin will flatten the estimated slope of the line from −3/2 toward −1, while inclusion of data points coming from stands of high density that have not yet begun to thin will steepen the slope of the line (Westoby 1984; Osawa and Allen 1993). Some authors considered that the maximum size–density relationships for given data sets were curvilinear instead of linear on a log–log scale (Zeide 1987; Cao et al. 2000). Others argued that it is possible that some plots at lower densities in a given data set may have not reached the stage of self-thinning yet (i.e., growth periods are not sufficiently long). Therefore, these plots should not be included for estimating the self-thinning line (Westoby 1984; Osawa and Allen 1993). In this study we assume the maximum size–density relationship is linear across the entire range of tree densities.

Over the last three decades researchers have applied different methods for selecting appropriate data points used to estimate the maximum size–density relationship. A common method is to purposefully select data points that lie close to an arbitrarily visualized upper boundary based on some criteria (Westoby 1984; Osawa and Sugita 1989; Osawa and Allen 1993; Wilson et al. 1999). As many authors have pointed out, this method is arbitrary and subjective. To improve selection objectivity, Bi and Turvey (1997) plotted the stand biomass (y-axis) against density (x-axis) on a log–log scale and divided the range of log density into a specified number of equal intervals. Then one data point of maximum stand biomass was selected from each interval. A similar method was used in an animal study (Blackburn et al. 1992). In a recent study, Solomon and Zhang (2002) assumed the theoretical value of the slope coefficient of the maximum size–density line (i.e., −1.5 for the $\log M \sim \log N$ relationship, where M is mean tree volume). The intercept coefficient was calculated by $a = \log M + 1.5 \log N$, using the stand with the largest relative density (RD) (Drew and Flewelling 1979). Once determined, the equation was used to compute maximum stand density (N_{max}) for M of a given stand. The RD was calculated as N/N_{max} for each stand, where N is current stand density. Stands with RD ≥0.7 were then selected for developing the maximum size–density relationship.

Historically, there are several regression methods used to estimate the two coefficients of the maximum size–density line, such as (1) arbitrarily hand fitting a line above an upper boundary of data points (Yoda et al. 1963; Drew and Flewelling 1977), (2) fitting an ordinary least squares regression (OLS) or weighted least squares regression (WLS) through data points selected on a density-dependent mortality criterion (Ford 1975; Wilson and Lee 1988), and (3) estimating coefficients via major axis analysis or principal components analysis (PCA) based on chosen data (Mohler et al. 1978; Hutchings and Budd 1981; Bi and Turvey 1997; Wilson et al. 1999). The last two methods define an "average" maximum size–density line (Osawa and Sugita 1989) rather than a "biological" maximum size–density line (Smith and Woods 1997). In theory, the maximum size–density line should be the upper boundary line of the selected data points (Weller 1987). Many analyses and applications of the −3/2 power law have failed to account for the asymptotic and limiting nature of the maximum size–density line when estimating coefficients. To correct this problem, Solomon and Zhang (2002) shifted the "average" maximum size–density line estimated by reduced major axis (RMA) regression parallel and upward to intersect the plot with the largest RD by increasing the estimated intercept, while preserving the estimated slope value.

In a "concept" paper in *Ecology*, Thomson et al. (1996) discussed the inappropriateness of commonly used standard statistical methods such as correlation and regression for estimating or testing limiting relationships in ecology. They suggested alternative methods for estimating functions along the edges of distribution such as mixture models (Maller 1990; Kaiser et al. 1994) and production frontiers (Färe et al. 1994). Alternatively, Scharf et al. (1998) and Cade et al. (1999) applied quantile regression to model-limiting relations and account for unmeasured ecological factors by estimating changes near upper extremes of data distributions. Bi et al. (2000) and Bi (2001, 2004) adopted a stochastic frontier production function to estimate the self-thinning line for even-aged pure pine stands, concluding that it used all data points without subjective selection and provided an efficient estimation of the self-thinning upper boundary.

The purpose of the present study is to compare alternative regression methods that have been or can be used to estimate the self-thinning boundary line. The methods include three regression-based techniques and three techniques used in economics studies (production functions). The self-thinning lines obtained by the six methods are also compared with traditional ways of fitting the line such as hand fitting the upper boundary line, fitting the line using the plots selected by the interval method, and fitting the line using the plots selected based on RD. An example was used to demonstrate and compare the methods. However, we emphasize that the objective of this study is not to test whether or not the slope coefficients estimated by alternative methods are significantly different from the theoretical constant.

Theoretical background

First we briefly review the three regression-based techniques: OLS, RMA regression, and quantile regression (QR).

OLS

OLS is generally acknowledged to be the best method for estimating the conditional mean of one random variable given a fixed value of another. For estimating the two coefficients of the maximum size–density line, however, researchers have realized that OLS is inappropriate because in this case the primary interest is in the values of the equation parameters themselves, which are used to describe the functional relationship between two random variables (Leduc 1987).

RMA

Mohler et al. (1978) introduced the use of PCA to estimate the regression coefficients between log$M$ and log$N$. They argued that PCA is preferable to OLS, since the first axis represents the line that minimizes the sum of the squared perpendicular distances of the points to the line and thus makes no assumptions about which is the dependent and which is the independent variable. Within the techniques of PCA, two variants are commonly used: major axis regression (MA) and RMA. One limitation of MA is that it is sensitive to the scale of measurement and can yield different results when the axes of the bivariate plot are rotated. RMA, on the other hand, overcomes the scale dependence of the MA regression technique by standardizing the variables M and N before the scaling exponent is computed (LaBarbera 1989; Niklas 1994). RMA can be summarized as follows: Assume a linear regression model $y = \alpha + \beta x + \varepsilon$, where y and x are dependent and independent variables, respectively, α and β are OLS regression coefficients, and ε is a random error term. The RMA slope coefficient is $\beta_{RMA} = \beta / |r_{yx}|$, where r_{yx} is Pearson's correlation coefficient between y and x. The standard error (SE) of β_{RMA} is equal to the SE of β. The RMA intercept coefficient is $\alpha_{RMA} = \bar{y} - \beta_{RMA}\bar{x}$, and the SE of α_{RMA} is equal to the SE of α (Solomon and Zhang 2002).

QR

Classic OLS regression can be viewed as a natural way of estimating conditional means for modeling central tendency. In contrast, QR solves an optimization problem of minimizing an asymmetric function of absolute error loss (Koenker and Bassett 1978; Koenker and Portnoy 1996; Bi et al. 2002). It is capable of providing statistical analysis and estimation for linear model fit to any part of a response distribution, including near the upper bounds, without imposing stringent assumptions on the error distributions (Scharf et al. 1998; Cade et al. 1999; Cade and Noon 2003). The τth quantile ($0 \le \tau \le 1$) of a random variable y is defined as the smallest real value of y such that the probability of obtaining any smaller values is greater than or equal to τ. For a linear model $y = X\beta + v(X)\varepsilon$, the τth regression quantile of y conditional on X is defined as $Q_y(\tau|X) = X\beta(\tau)$, where $\mathbf{y}$ is an $n \times 1$ vector of dependent responses, $\mathbf{X}$ is an $n \times p$ matrix of predictors (the first column of $\mathbf{X}$ consists of 1's (an intercept)), β is a $p \times 1$ vector of unknown regression coefficients, $v(\cdot) > 0$ is a known function, and ε is an $n \times 1$ vector of random errors. The $\beta(\tau)$ can be estimated by minimizing an asymmetric loss function of absolute values of residuals, where positive residuals are given weights equal to τ and negative residuals are given weight equal to $1 - \tau$ as follows:

$$\min\left[\sum_{i\in\{i|y_i\ge\hat{\beta}_j x_{ij}\}}\tau\left|y_i-\sum_{j=0}^{p}\hat{\beta}_j x_{ij}\right|+\sum_{i\in\{i|y_i<\hat{\beta}_j x_{ij}\}}(1-\tau)\left|y_i-\sum_{j=0}^{p}\hat{\beta}_j x_{ij}\right|\right]$$

Clearly, positive and negative residuals are differentially weighted for QR other than at $\tau = 0.5$. The advantages of QR are (1) it is robust to distribution assumptions, (2) it is equivalent to monotonic transformation, (3) the term $v(\mathbf{X})$ allows the random error ε to change as a function of $\mathbf{X}$, accommodating both homogeneous and heteroscedastic error models, and (4) QR estimates are insensitive to extreme values of outlying dependent variables. However, the variance of quantile regression estimator is U-shaped with τ changing from zero to one. Thus, the estimated boundary line by quantile regression can be variable even with a very small change in τ. The variability is particularly high when the number of data points under analysis is small (Scharf et al. 1998; Cade and Noon 2003).

Next we review three methods for estimating frontier production functions. In economics, a frontier production function is defined as a function giving the maximum possible output for a given input set and can be used to study technical efficiency of individual firms. The word "frontier" emphasizes the idea of maximality (Schmidt 1985–1986; Greene 1997). Let a production process or technology be represented by $Q_i = f(x_i, \beta)e^{\varepsilon_i}$, where Q denotes output, x denotes a set of inputs, β is a set of parameters to be estimated, and i denotes producers. In most applications, after transformation, a production function is linear in the logarithmic scales of output and a set of independent variables:

$$y_i = \log Q_i = \alpha + \beta x_i + \varepsilon_i$$

Since a production function gives maximal output rather than mean output by definition, the residuals ε_i are assumed to possess a nonzero mean and constant variance and to be randomly distributed across firms (Greene 1997). Econometric researchers have developed many methods for estimating frontier production functions. We review three of these next.

Corrected OLS (COLS)

In all frontier production functions, the slope parameter β can be consistently estimated by OLS, since OLS is robust to non-normality. The intercept parameter α in a frontier model can be consistently estimated simply by shifting the OLS line upward so that the largest residual is zero. That is

$$\alpha_{COLS} = \alpha + \max_i \varepsilon_i$$

This procedure is known as COLS (Greene 1993, 1997). Kopp and Mullahy (1993) showed that COLS was actually a method of moments estimator and provided expressions for

the variances. They further concluded that COLS was an appealing alternative to maximum likelihood (ML) estimation of frontier models because of its computational simplicity and relative robustness.

Deterministic frontier function (DFF)

In a deterministic frontier model, output is bounded from above by a deterministic (nonstochastic) production function. Aigner and Chu (1968) used linear programming and quadratic programming to fit a DFF. In the linear programming approach, the sum of the absolute values of the residuals are minimized as

$$\min_{\beta} \sum_i |y_i - \alpha - \beta x_i|$$

$$\text{subject to } \varepsilon_i = y_i - \alpha - \beta x_i \le 0, \forall i$$

whereas in the quadratic programming approach, the sum of squared residuals is minimized as

$$\min_{\beta} \sum_i (y_i - \alpha - \beta x_i)^2$$

$$\text{subject to } \varepsilon_i = y_i - \alpha - \beta x_i \le 0, \forall i$$

The negative residuals force all observations of output to be on or below the frontier function through a set of constraints. However, the problems arising with the mathematical optimization approach include (1) undue sensitivity to outliers, (2) lack of SE for the estimated parameters, and (3) statistics for inference is difficult (Greene 1980).

Stochastic frontier function (SFF)

In contrast, a stochastic frontier model specifies that the maximum output a producer can obtain is assumed to be determined both by the production function and by random external factors (Aigner et al. 1977; Greene 1993, 1997; Kumbhakar and Lovell 2000). An appropriate model for the stochastic frontier is

$$\begin{aligned} y_i &= \log Q_i \\ &= \alpha + \beta x_i + \varepsilon_i \\ &= \alpha + \beta x_i + v_i - u_i \end{aligned}$$

where $\varepsilon_i = v_i - u_i$ is a compound error term with $u_i \ge 0$ and v_i unrestricted. Both components of the compound error term are generally assumed to be independent and identically distributed (i.i.d.) across observations. The v_i are usually assumed to have a symmetric distribution such as a normal distribution, that is, $v_i \sim N(0, \sigma_v^2)$ and represent any stochastic factors beyond the firms' control. A stochastic frontier model collapses into a deterministic frontier model when $\sigma_v^2 = 0$. The u_i embody the one-side (asymmetric) part of the compound error term ε_i (Aigner et al. 1977; Greene 1993, 1997). Several specifications have been considered for u_i: (1) a half-normal distribution $u_i \sim |N(0, \sigma_u^2)|$ (Aigner et al. 1977) — in this case, $E(u_i) = (\sqrt{2/\pi})\sigma_u$ and $\text{var}(u_i) = (1 - 2/\pi)\sigma_u^2$; (2) an exponential distribution $f(u_i) = \theta e^{\theta} u$ with $\theta > 0$ and $u_i > 0$ (Aigner et al. 1977); and (3) a truncated normal (Stevenson 1980). The one-side error component u_i is taken to be a variable obtained by truncating at zero with a possibly nonzero mean, that is, $u_i \sim N(\mu, \sigma_u^2)$. The three aforementioned specifications were implemented in LIMDEP computer software (Greene 1998).

Materials and methods

In this study an example stand of even-aged eastern white pine (*Pinus strobus* L.) was used to compare the alternative methods for estimating the maximum size–density relationship of self-thinning. A total of 262 permanent plots of white pine (constituting ≥80% of total basal area) were obtained from the database used for the development of FIBER 3.0 (Solomon et al. 1995). The descriptive statistics of variables are listed in Table 1. The RD was computed for each plot based on the relationship between log$D$ and log$N$ (i.e., assuming the slope coefficient = –0.60), following a procedure similar to the one in Solomon and Zhang (2002).

We decided to work with the relationship between log$D$ and log$N$ rather than that between log$M$ and log$N$ for several reasons: (1) the quadratic mean tree diameter (D) is a common stand variable used in forestry practice, and it is a direct measurement of average tree size in forest inventory, (2) there is an exact relationship between D, N, and stand total basal area, and (3) we avoid introducing biases into the analysis caused by computing tree volume from a specific volume equation or table (Curtis and Marshall 2000). The regression model was

$$[1] \qquad \log D = \beta_0 - \beta_1 \log N + \epsilon$$

where β_0 and β_1 are regression coefficients to be estimated and ϵ is a model error term.

Firstly, we purposefully selected two data points that lay close to a visualized upper boundary for all available plots. The β_0 and β_1 coefficients were calculated based on the x (i.e., log$N$) and y (i.e., log$D$) coordinates of the two plots (namely hand-fitting method). Secondly, the range of log$N$ was divided into equal intervals, and one plot with the maximum log$D$ was selected from each interval. These plots were used to fit eq. 1 by OLS (namely interval method) (Scharf et al. 1998; Bi and Turvey 1997). Thirdly, the plots with RD ≥ 0.85 were used to fit eq. 1 by OLS (namely RD method) (Solomon and Zhang 2002). Lastly, all available plots (n = 262) were used to fit eq. 1 by the six regression methods reviewed in the last section. SAS (SAS Institute Inc. 1999) was used for the OLS, RMA, QR (regression quantile $\tau = 0.999$), and COLS methods. LINDO was used for DFF methods with the linear programming approach (LINDO System Inc. 1998), and LIMDEP 7 was used for SFF methods assuming u_i follows a half-normal distribution (Econometric Software Inc. 1998).

Results and discussion

In the hand-fitting method a visualized line was placed across the upper boundary of available plots (Fig. 1). The x and y coordinates of the top-most two plots ($\log D_1$ = 3.26919, $\log N_1$ = 7.15851, and $\log D_2$ = 3.66587, $\log N_2$ = 6.42649) were used to compute the β_0 and β_1 coefficients of eq. 1, resulting in

$$[2] \qquad \log D = 7.15 - 0.54 \log N$$

Table 1. Descriptive statistics of variables (n = 262 plots).

Variable	Mean	SD	Min.	Max.
Quadratic mean diameter (cm)	26.1	8.3	13.8	64.7
Density (trees/ha)	726	439	17	2619

Although the hand-fitting method is considered arbitrary and subjective, the resultant self-thinning line is the upper limiting boundary for the maximum size–density relationship (Yoda et al. 1963; Drew and Flewelling 1977; Niklas 1994).

Following the interval method in Bi and Turvey (1997), the range of log$N$ was divided into seven equal intervals (Fig. 2). One plot with the maximum log$D$ was selected from each interval. These seven plots were used to fit eq. 1 by OLS and produced the following model:

$$[3] \quad \log D = 6.50 - 0.46\log N$$

More intervals (12 intervals) were also tried, but the resultant model (β_0 = 6.43 and β_1 = –0.45) was very similar to eq. 3. The interval method is less subjective than the hand-fitting method. However, it often generates a small sample size for model fitting (Blackburn et al. 1992; Bi and Turvey 1997), and the coefficient estimates may also vary depending on the number of size classes and the method of dividing size classes (Scharf et al. 1998). More importantly, it is possible to include some plots at lower densities (i.e., at the left end of the log$N$ axis) that have not reached the stage of self-thinning. Consequently, the slope coefficient of the self-thinning line based on this subset of the plots may be flatter than expected (Westoby 1984; Osawa and Allen 1993). In this case, the slope coefficient (–0.43) of eq. 3 would produce a line flatter than the line defined by eq. 2 (β_1 = –0.54).

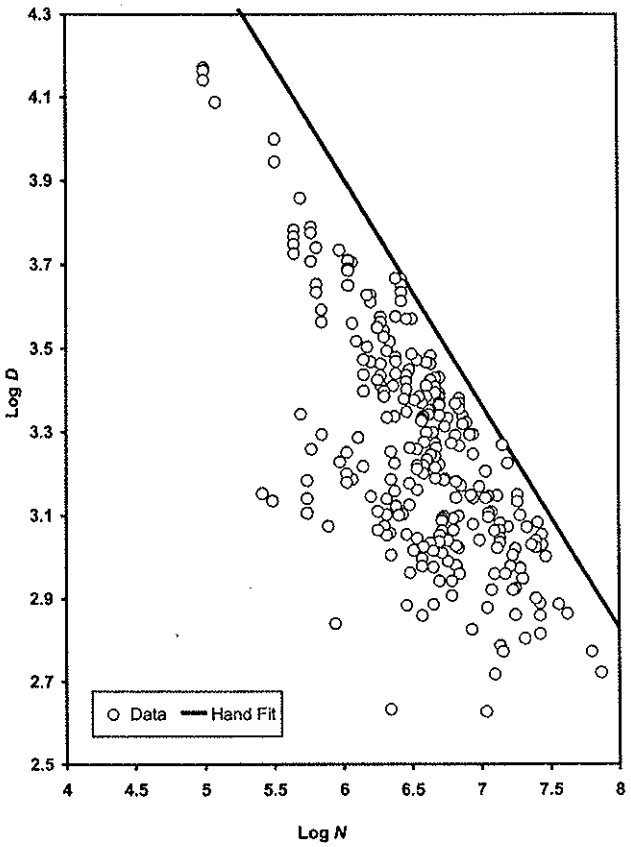
Solomon and Zhang (2002) considered any plot with a high RD (say RD ≥ 0.70) to be undergoing a self-thinning process and experiencing density-related mortality. Therefore, it is reasonable to select plots with an RD larger than a predetermined threshold value to fit the maximum size–density relationship. In this study we chose 0.85 as the threshold value for RD and ended up with 21 plots (Fig. 3). The resultant OLS model was

$$[4] \quad \log D = 7.66 - 0.62\log N$$

Using a higher threshold value of RD (e.g., RD ≥ 0.90) produced a similar model with β_0 = 7.62 and β_1 = –0.61. The concern with the RD method is that the calculation of RD for each plot is based on a theoretical constant for the slope coefficient in eq. 1 (i.e., –0.6 in this study). Thus, the central tendency of this subset of the plots has been predetermined or influenced by the theoretical slope constant. In this case, the estimated slope (–0.62) by the RD method was close to the theoretical constant, and much steeper than those of eqs. 1 and 2.

One way to avoid subjectively selecting data points is to use all available plots and fit the self-thinning line by appropriate regression techniques. Next we focus on the comparison of the six regression methods reviewed in the Theoretical background section. Table 2 shows the two regression coefficients for the six models. Figure 4 illustrates

Fig. 1. The maximum size–density line obtained from the hand-fitting method.



the regression lines obtained by the six modeling methods. It was clear that OLS represented a central tendency line (β_0 = 5.78 and β_1 = –0.38) across the range of data. The COLS method moved the OLS line upward to intersect the plot with the largest OLS residual $\hat{\varepsilon}$ = (0.35). The COLS method increased the estimate of the intercept coefficient from 5.78 to 6.13, while preserving the estimate of the slope value (–0.38). However, the COLS line was not appropriate to describe the maximum size–density relationship because of the inappropriate slope coefficient of the OLS line. The two coefficients (β_0 = 6.86 and β_1 = –0.55) of the RMA line were recalculated based on Pearson's correlation coefficient between log$D$ and log$N$ (Solomon and Zhang 2002). The result was an "average" line across the data. If the COLS method was used again to "correct" the RMA line given the plot with the largest OLS residual, the new intercept coefficient would be 7.21 instead of 6.86, while the slope coefficient remained the same (–0.55). It would produce a line (not shown in Fig. 4) close to the QR and DFF lines discussed next.

QR and DFF resulted in the same intercept (β_0 = 7.15) and slope (β_1 = –0.54) coefficients for eq. 1. Since both QR and DFF methods forced all observations to be on or below a limiting boundary line, they produced a self-thinning line very similar to the upper limiting boundary line of the hand-fitting method (eq. 2).

On the other hand, the error term ε_i in the SFF method has an asymmetric and non-normal distribution with a nega-

Fig. 2. The maximum size–density lines obtained using the plots selected by the interval method, using ordinary least squares (OLS).

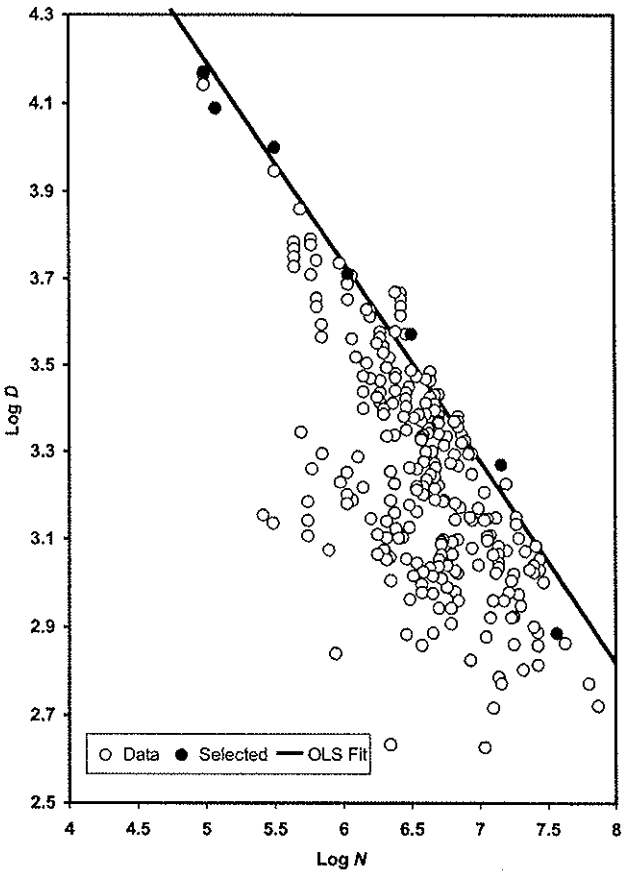
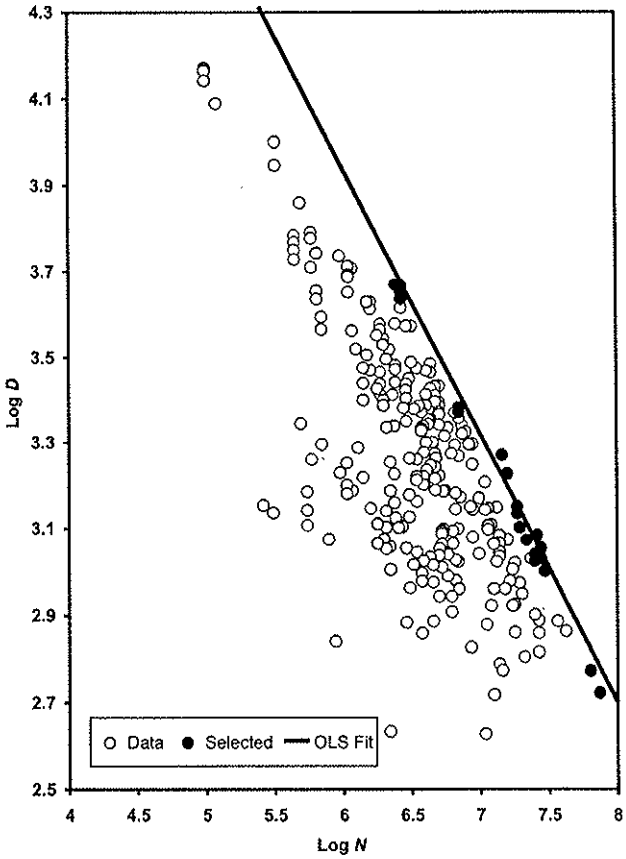


Fig. 3. The maximum size–density lines obtained using the plots selected with RD > 0.85 and fit using ordinary least squares (OLS).



tive mean. But a negative mean does not imply that all residuals are negative and allows a few residuals to be positive, especially when σ_v^2 is much larger than zero (Bi et al. 2000). Therefore, the SFF method yielded a maximum size–density line ($\beta_0 = 6.47$ and $\beta_1 = -0.45$) "lower" than the upper limiting boundary line (Fig. 4). This line describes the maximum size–density relationship by taking into account site-occupancy due to density-dependent growth and mortality within individual stands and the effects of external factors that take place at random over space and time on the frontier (Guo and Rundel 1998; Bi et al. 2000; Bi 2001). Its intercept and slope are similar to those of eq. 3 by the interval method. Such comparability was consistent with other studies (Bi and Turvey 1997; Bi 2000). However, the SFF method can yield an upper limiting boundary line only when the estimated σ_v^2 is small and close to zero, as in the case of Bi (2004).

Summary

Our results indicate that QR, DFF, SFF methods have the potential to produce an upper limiting boundary line above all plots for the maximum size–density relationship, without subjectively selecting a subset of data points based on predefined criteria. In contrast, OLS, COLS, and RMA methods are sensitive to the data selected for model fitting and may produce self-thinning lines with inappropriate slopes. However, statistical inference is very difficult with DFF and QR methods. Although SFF produces a self-thinning line lower than the upper limiting boundary line because of the nature of the method, the method can easily perform statistical inference on the model coefficients, given that there are no significant departures from underlying distributional assumptions.

Table 2. Regression coefficients of the six models.

Method	β_0	β_1
Ordinary least squares (OLS)	5.78	–0.38
Reduced major axis (RMA)	6.86	–0.55
Quantile regression (QR)	7.15	–0.54
Corrected ordinary least squares (COLS)	6.13	–0.38
Deterministic frontier function (DFF)	7.15	–0.54
Stochastic frontier function (SFF)	6.47	–0.45

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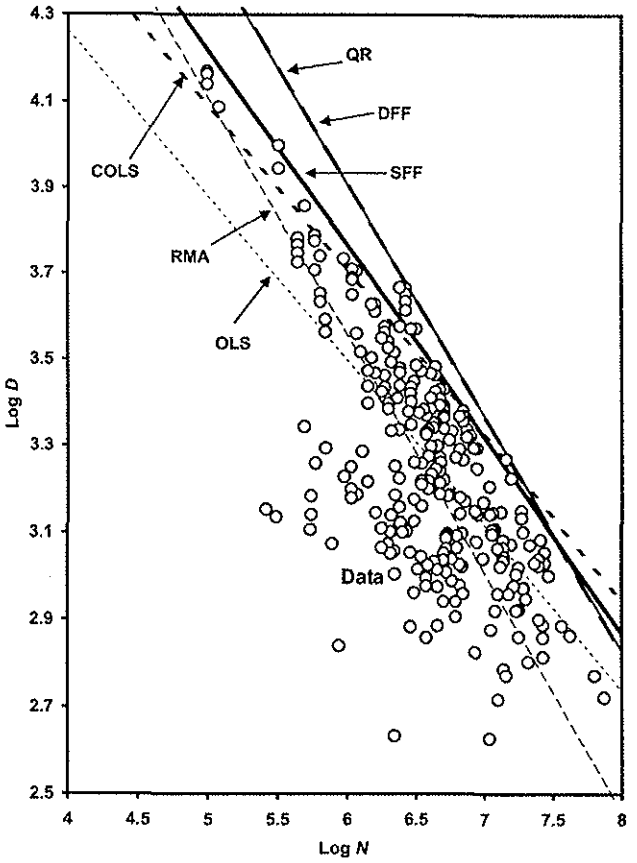
Fig. 4. The maximum size–density lines obtained from the six modeling methods. Corrected ordinary least squares, COLS; *deterministic frontier function, DFF; ordinary least squares, OLS*; QR, quantile regression; reduced major axis, RMA; stochastic frontier function, SFF.



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