Mathematical Form Models of Tree Trunks

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Abstract.—Assortment structure analysis of tree trunks is a characteristic and proper problem that can be solved by using mathematical modeling and standard computer programs. Mathematical form model of tree trunks consists of tapering curve equations and their parameters. Parameters for nine species were obtained by processing measurements of 2,794 model trees and studying the variability of 53,000 relative diameters.

Tapering tables comprise a large number of tree trunk dimensions, which are not convenient to handle even by advanced computers. The idea of taking advantage of empirical formulas as a compact storage of information is not new. A project for the creation of mathematical tapering table models was accepted at the Latvian Forest Inventory Institute 25 years ago. Basically this research work was completed in 1982. At present, investigations are continuing in different directions. A program for producing variable assortment tables was developed by the author of this paper and approved at the Latvian Forestry Research Institute "Silava." During the last few years, diverse simplified versions of variable tables have been applied in Latvia as official state standards.

METHODS

Methods used for this study can be divided into two groups. The first group includes some widely used statistical methods. The second group contains original, not ordinary, methods: hyperbolical interpolation, designing of empirical formula by linear transformation, perturbation of relative diameters, generation of variable assortment tables, designing of round wood volume formulas, and other methods. These methods can be useful in forest research.

Hyperbolical Interpolation

The intention to use the equation of hyperbola for smoothing the connection between the thickness and height of a tree for the first time was approbated in 1968. Advantages of hyperbola as compared to classical parabolical interpolation for the approximation of monotonous values proved to be obvious. Measurements of model trees are necessary information for designing the tapering curve equation. Rough data were collected from different places in Latvia by different methods over several years. The first stage of primary information processing was the computation of actual diameters by using hyperbolical interpolation (form. 1):

$$d = d_1 + \frac{h - h_1}{k(h - h_1) + c}, \quad \text{where}$$

$$c = \frac{\dfrac{1}{d_2 - d_1} - \dfrac{1}{d_3 - d_1}}{\dfrac{1}{h_2 - h_1} - \dfrac{1}{h_3 - h_1}};$$

$$k = \frac{1}{d_2 - d_1} - \frac{c}{h_2 - h_1} \qquad (1)$$

The conventional signs used are:

h_1, h_2, h_3—distances from the butt-end to the given cut, m;

d_1, d_2, d_3—actual corresponding diameters, cm;

h—distance from the butt-end to a freely selected cut, m;

d—actual calculated diameter, cm.

If $h_1 < h_2 < h_3$, then h can be selected from the interval fixed by inequality $h_1 < h < h_3$. Extrapolation is not permitted.

Designing of Empirical Formulas by Linear Transformation

Actual diameters are usually estimated in different places of a tree trunk, beginning from the butt-end with a step of 2 m. In total, results of measuring can be displayed as the tapering curve or side-face of the model tree. For instance, if tapering curves of 15 birch trunks with diverse parameters or dimensions are drawn in one picture, a sufficient chaotic crossing of different broken lines can be obtained. It is impossible to notice any connection (fig. 1).

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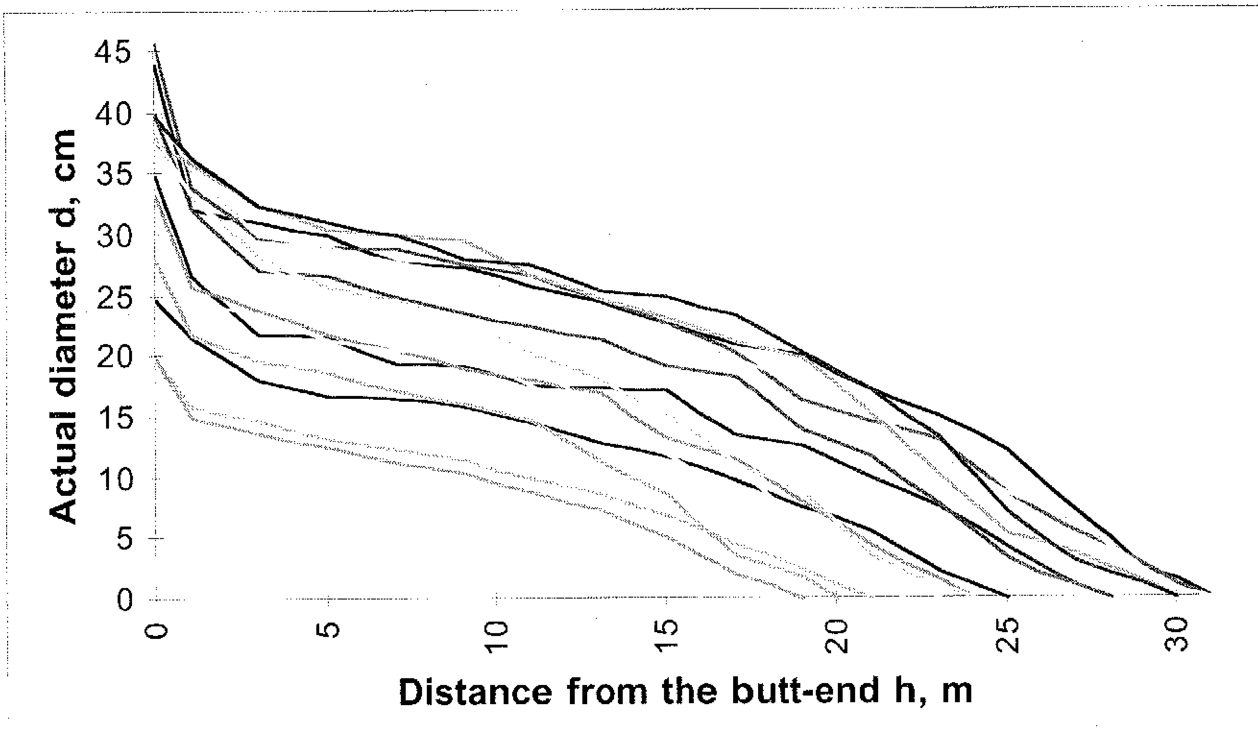

Figure 1.—*Tapering lines of 15 birch model trees before transformation.*

When looking for a solution, the recommendation was to divide the height of a tree into 20 identical parts and calculate corresponding actual diameters:

$$d_i; \quad i=0, 1, 2,..,19.$$

This was done by using hyperbolical interpolation (form. 1). The last diameter at the top of a tree is constantly equal to zero. Now it is possible to realize the linear transformation in longitudinal direction by changing the absolute height by the relative:

$$x = \frac{h}{H}, \text{ where} \qquad (2)$$

h—distance from the butt-end to a freely selected cut, m;
H—height of tree, m;
x—relative height.

In fact, the accomplished transformation in longitudinal direction is an even shortening of a trunk *H* times (fig. 2). To carry out a transformation in transverse direction or change the thickness of a trunk, the diameter in a definite relative height—for instance, 0.1—must be assumed equal to a constant value, for instance, 100:

$$\delta(x) = \frac{100 \cdot d(h)}{d(0.1H)}. \qquad (3)$$

The conventional signs used are:
d(h)—actual diameter in height *h*, cm;
d(0.1*H*)—actual diameter in height 0.1*H, cm*;
δ(*x*)—conformable relative diameter in relative height x.

Constantly $\delta_2 = \frac{100 \cdot d_2}{d_2} = 100$;

$$\delta_{20} = \frac{100 \cdot d_{20}}{d_2} = 0 \cdot$$

The processing of 2,794 model tree measurements resulted in the calculation of 53,000 relative diameters. Arithmetic mean values of relative diameters in connection with the relative height (table 1) are very important for further research work. Studies of the relative diameter variabilities of pine, fir, birch, aspen, and black alder disclosed an interesting detail: tapering curves of foliage trees used to have a second inflection point in the relative height of approximately 0.8. Further processing of information was considerably simplified after the transition from absolute to relative values was carried out. Now it is more convincing to indicate the type of connection being searched (fig. 3).

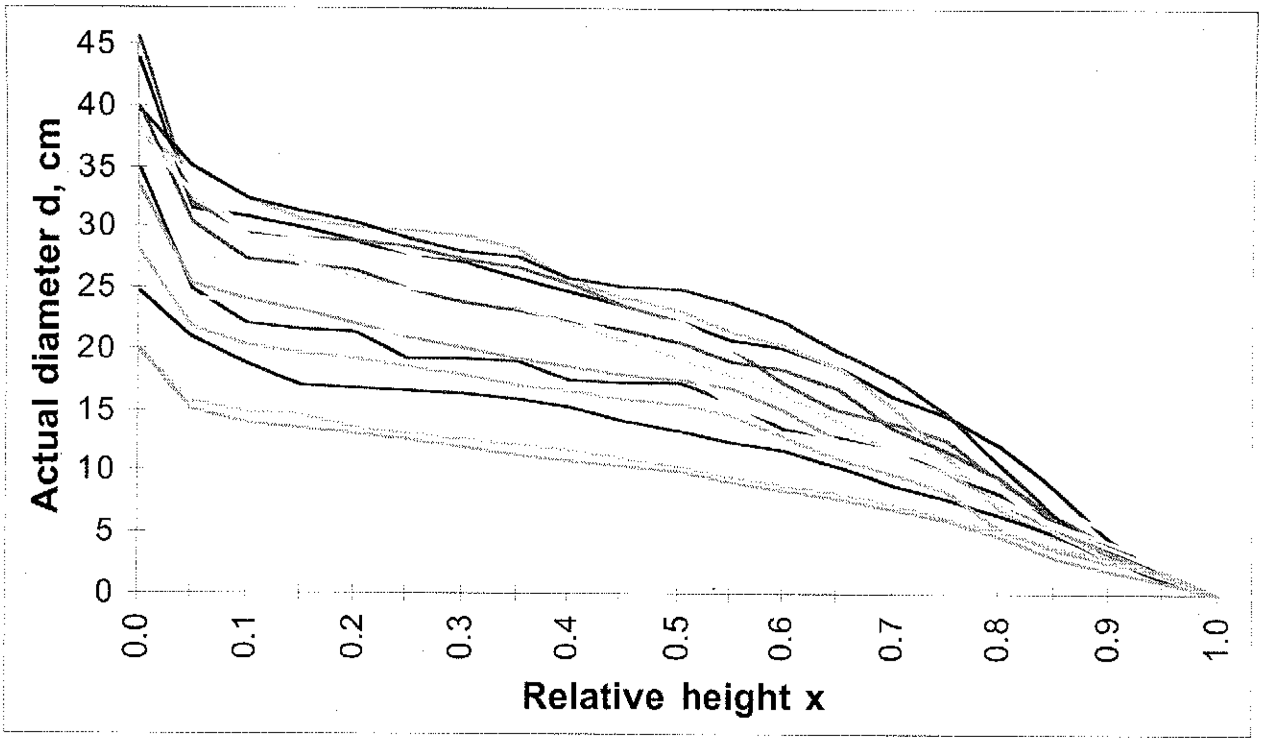

Figure 2.—*Tapering lines of 15 birch model trees after linear transformation in longitudinal direction.*

Table 1.—*Relative diameters of tree trunks*

Relative height	Pine	Fir	Birch	Aspen	Black alder	Relative height	Pine	Fir	Birch	Aspen	Black alder
0.00	129.5	132.3	137.5	120.2	124.7	0.55	68.8	68.3	65.1	67.1	67.2
0.05	107.6	105.6	108.8	104.4	107.5	0.60	64.8	63.3	59.5	61.7	62.4
0.10	100.0	100.0	100.0	100.0	100.0	0.65	60.3	57.9	53.7	55.8	57.0
0.15	94.8	95.9	94.9	95.9	95.5	0.70	55.2	52.1	47.3	48.8	51.0
0.20	91.1	93.0	91.6	93.2	91.7	0.75	49.1	45.8	40.2	41.5	43.3
0.25	87.8	90.3	88.2	90.4	88.6	0.80	42.0	38.6	32.8	33.6	34.8
0.30	84.7	87.4	85.0	87.1	85.6	0.85	33.4	31.1	24.7	24.8	26.2
0.35	81.8	84.1	81.5	83.3	82.2	0.90	23.5	22.9	17.0	16.5	17.9
0.40	78.8	80.7	77.7	79.3	78.8	0.95	12.3	14.3	10.0	9.6	9.5
0.45	75.7	77.0	73.9	75.3	75.4	1.00	0.0	0.0	0.0	0.0	0.0
0.50	72.4	72.8	69.8	71.5	71.5						

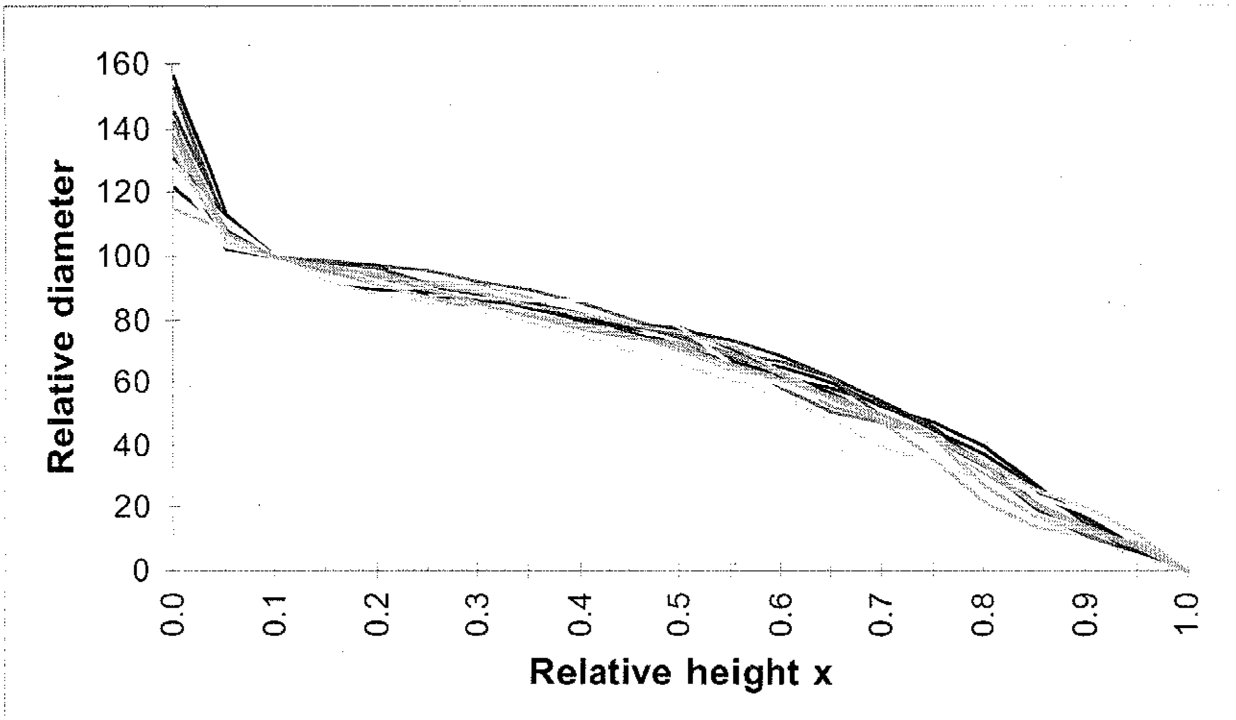

Figure 3.—*Tapering lines of 15 birch model trees after complete linear transformation in longitudinal and transverse direction.*

If the analytical expression for empirical formula is selected, a smoothing of relative diameters in dependence from the relative height (table 1) can be realized on computer by applying a standard program, for example, Microsoft Excel. Calculations have proven that the sixth power polynomial is good enough for ordinary cases. In accordance with the available information, coefficients of the sixth power polynomial

$$P_6(x) = a_0 + a_1x + a_2x^2 + \ldots + a_6x^6 \ , \qquad (4)$$

were calculated for nine tree species by using various methods in dependence of primary information (table 2). The average values of relative diameters can be approximated by empirical formula:

$$\delta = P_6(x). \qquad (5)$$

The final connection between a freely selected distance from the butt-end and the corresponding actual diameter for tree trunks with given dimensions can be obtained by applying reverse transformation, i.e., by an even extension or reduction of the mean transformed model of the tapering curve (form. 5) in longitudinal and transverse directions:

$$d = D \cdot \frac{P_6\left(\frac{h}{H}\right)}{P_6\left(\frac{1.3}{H}\right)}, \text{ where} \qquad (6)$$

D—diameter at breast height (1.3 m), cm.

The designed equation (form. 6), together with coefficients (table 2), is the mathematical model of tapering tables for tree trunks in bark. This model can be used for solving diverse tasks of forest mensuration on the computer.

Perturbation of Relative Diameters

Common features of the form of tree trunks for given species are included in the mean tapering curve equation (form. 6, table 2). An additional connection between relative diameters and dimensions of a trunk was confirmed after a more detailed investigation. It turned out that the choice of a suitable analytical expression for the approximation of the named values by using classical methods of mathematical statistics is a very complicated task. For reliable calculation results, several hundreds of model trees are needed. Because of the lack of sufficient customary measurements, an original access is necessary. Eventually, fluctuations of the form of a tree trunk are determined by biological features of species and by growing conditions. All forest forming trees have much in common. Typical features of a trunk with a given length and thickness can be obtained by changing the relative diameters within limits and specifying deviation from the average form by using indirect data such as

Table 2.—*Coefficients of the sixth power polynomial*

Tree species	a_0	a_1	a_2	a_3	a_4	a_5	a_6
Pine	118.981	-277.578	1140.525	-3037.487	4419.682	-3361.780	997.657
Fir	113.939	-203.061	827.209	-2161.251	2732.076	-1699.667	390.755
Birch	120.567	-312.074	1388.288	-3725.819	5197.005	-3788.858	1120.891
Black alder	120.224	-310.985	1450.125	-4238.703	6644.011	-5408.312	1743.640
Aspen	110.428	-143.288	530.481	-1643.304	2606.605	-2212.940	752.018
White alder	118.560	-263.482	988.135	-2376.874	3045.214	-2137.684	626.131
Oak	120.958	-354.769	2022.206	-6736.346	11231.250	-9254.632	2971.333
Ash	117.999	-282.941	1411.064	-4542.395	7964.660	-7175.007	2506.620
Lime	110.428	-143.287	530.477	-1643.287	2606.569	-2212.906	752.006

volume tables. Specification of deviation and calculation of parameters are based on their random and gradual change with succeeding computing of the volume of a tree trunk by the integration of the tapering curve equation. The final values of parameters correspond to the smallest difference between the volumetric table and the calculated results. The recommendation was to specify the deviation by using a particular coefficient called perturbation factor:

$$\gamma = 1 + \left(x^2 - 0.01\right) \cdot \left(p \cdot (H - H_0) + q \cdot (D - D_0)\right). \quad (7)$$

The conventional signs are:

D_0, H_0—dimensions of mean form trunk;
p, q—parameters of tree species.

Parameters were calculated for pine, fir, birch, black alder, aspen, white alder, oak, ash, and lime (table 3). If the perturbation factor (form. 7) is taken into account, the relative diameters are determined by formula:

$$\delta = \gamma \cdot P_6(x), \text{ or symbolically,}$$

$$\delta = F(\tfrac{h}{H}, D, H). \quad (8)$$

The equation of the trunk tapering curve becomes more complicated:

$$d = D \cdot \frac{F(\frac{h}{H}, D, H)}{F(\frac{1.3}{H}, D, H)}. \quad (9)$$

The variability of the tapering curve in connection with tree dimensions (fig. 4) is better reflected as compared to the mean values.

This perturbation method of relative diameters may seem strange. But precision is the main criterion for any method. It is not complicated to collect additional data and test recommendations. The use of a perturbation factor has allowed the design of a uniform system of standards for the assortment structure analysis in felling areas before cutting.

Variable Assortment Tables

To carry out diverse calculations of roundwood volume, a law of bark thickness distribution is needed. Analysis of model tree measurements has revealed a sufficiently exact connection between the relative thickness of bark and the relative height of the given cut. The fourth power polynomial was recommended for the approximation of the above named connection:

$$p = Q_4(x),$$

$$Q_4(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4, \quad (10)$$

Table 3.—*Parameters of perturbation factor*

Parameters	Pine	Fir	Birch	Black alder	Aspen	White alder	Oak	Ash	Lime
H_0	26	33	20	14	18	16	14	21	16
D_0	30	36	28	12	30	16	20	20	12
p	0.0070	0.0087	0.0210	0.0264	0.0074	0.0168	0.0263	-0.0021	0.0061
q	-0.0070	-0.0197	0.0000	-0.0017	0.0002	-0.0103	0.0005	0.0000	0.0000

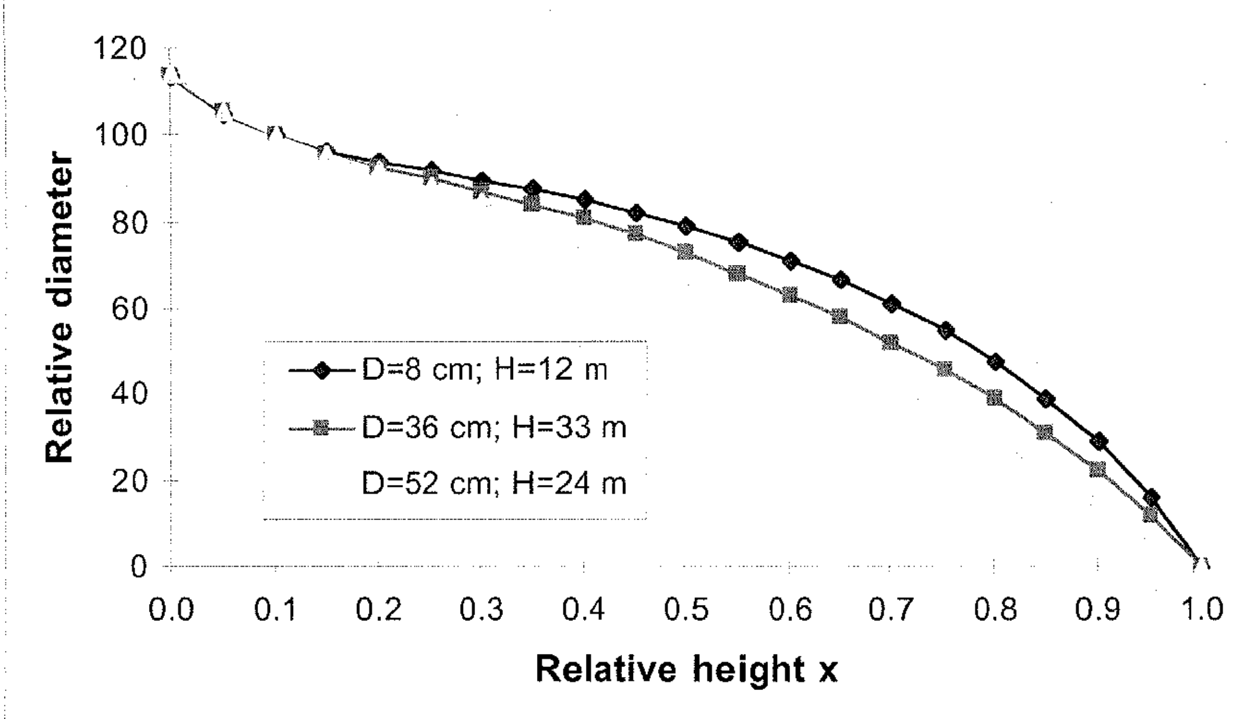

Figure 4.—*Changes of fir trunk tapering lines in dependence from tree dimensions.*

The conventional signs are:

- p—double thickness of bark in percent from the diameter in bark,
- x—relative height,
- b_0, b_1, b_2, b_3, b_4—coefficients of the fourth power polynomial.

Coefficients were calculated for pine, fir, birch, black alder, aspen, white alder, oak, ash, and lime (table 4). Considering the equation of the tapering curve (form. 6) and the law of bark distribution (form. 10), an equation of a tapering curve for a tree trunk without bark was obtained:

$$d^* = D \cdot \frac{P_6\left(\frac{h}{H}\right)}{P_6\left(\frac{1.3}{H}\right)} \cdot \left[1 - \frac{Q_4\left(\frac{h}{H}\right)}{100}\right], \text{ where} \qquad (11)$$

d^*—diameter of the trunk without bark at a freely selected height, cm.

A more exact result is obtained if the equation with perturbation factor (form. 9) and the law of bark distribution (form. 10) are taken into consideration:

Table 4.—*Coefficients of the fourth power polynomial*

Coefficients	Pine	Fir	Birch	Black alder	Aspen	White alder	Oak	Ash	Lime
b_0	12.90	5.35	9.61	8.34	7.57	5.02	12.81	8.31	8.00
b_1	-55.16	-7.57	-39.92	0.93	-17.99	-10.66	-53.23	-19.75	0.89
b_2	116.32	20.37	117.49	20.45	43.35	25.69	156.65	47.60	19.61
b_3	-110.12	-13.10	-134.22	-62.45	-37.07	-21.97	-178.96	-40.70	-59.89
b_4	40.96	5.45	55.73	55.00	14.24	8.44	74.31	15.64	52.75

$$d^* = D \cdot \frac{F\left(\frac{h}{h}, D, H\right)}{F\left(\frac{1.3}{H}, D, H\right)} \cdot \left[1 - \frac{Q_4\left(\frac{h}{H}\right)}{100}\right]. \qquad (12)$$

On average, the improved tapering curve equation without bark (form. 12) conforms more closely to the form of the given tree trunk. However, calculation by that formula is more complicated.

The roundwood, the quality of which corresponds to the conditions of standards, is usually called the assortment. The volume of roundwood (or log) including bark can be calculated by the integration of the tapering curve (form. 6):

$$v = \frac{\pi}{4 \cdot 10^4} \cdot \int_{h_1}^{h_2} \left[D \cdot \frac{P_6\left(\frac{h}{H}\right)}{P_6\left(\frac{1.3}{H}\right)} \right]^2 dh \text{ , where} \qquad (13)$$

h_1, h_2—distances from the butt-end to the beginning and end of a log, m;
v—volume of log in bark, cubic meter.

For the first log, h_1=0, h_2 - length of log.

If h_1=0 and h_2=H, the volume of tree trunk V is obtained:

$$V = \frac{\pi}{4 \cdot 10^4} \cdot \int_0^H \left(D \cdot \frac{P_6\left(\frac{h}{H}\right)}{P_6\left(\frac{1.3}{H}\right)} \right)^2 dh. \qquad (14)$$

After the elementary alteration formula (form. 14) becomes more convenient for computation:

$$V = \frac{\pi \cdot D^2 \cdot H}{4 \cdot 10^4 \cdot \left[P_6\left(\frac{1.3}{H}\right)\right]^2} \cdot I_6 \text{ , where}$$

$$I_6 = \int_0^1 \left[P_6(x) \right]^2 dx \; . \qquad (15)$$

If the volume of roundwood without bark is calculated, the tapering curve equation (form. 11 or 12) can be used. If, for example, the improved equation with perturbation factor (form. 12) is chosen, then

$$v^* = \frac{\pi \cdot D^2}{4 \cdot 10^4 \cdot F^2\left(\frac{1.3}{H}, D, H\right)} \cdot \int_{h_1}^{h_2} F\left(\tfrac{h}{H}, D, H\right) \cdot \left(1 - \frac{Q_4\left(\frac{h}{H}\right)}{100}\right)^2 dh, \text{ where} \qquad (16)$$

v*—volume of log without bark, cubic meter.

Now the required tools are ready for the generation of variable roundwood tables: the necessary equations (form. 6, 9, 11, 12) have been designed, and the uniform parameters (tables 2, 3, 4) have been calculated. Theoretically, it becomes possible to imaginarily saw the growing tree trunk in diverse versions. For each variant, a volume of roundwood, either including bark or not, can be computed. There may be some condition such as agreement with the wood processor or buyer, quality of wood, priority of fixed dimensions. The final results of the described calculations form an ordinary version of variable assortment tables. The first trials to compile variable tables were carried out at the Latvian Forest Research Institute 15 years ago.

An empirical formula has become a convenient and compact storage of information for advanced computers. After choosing an analytical expression, parameters can be obtained by mathematical simulation. For instance, the volume without bark for veneer timber sawed from the butt-end part of a birch stem can be calculated by formula:

$$V = EXP(-9.63316) \cdot L^{0.884752} \cdot R_{05}^{2.09172} \cdot T_0^{-0.0909147}, \text{ where} \qquad (17)$$

L—length of veneer timber, m;
R_{05}—diameter (with bark) at 0.5 m from the butt-end in centimeters;
T_0—diameter (with bark) at the thin end of veneer timber in centimeters.

DISCUSSION

For sustainable management of forests a variety of information about the resources is needed. Some problems need further investigation. Certainly the collection of data is a very labor consuming process, but it is necessary for improving existing mathematical models or for creating a new generation of advanced models. Several directions are possible for further study. The diameter of a tree trunk at 1.3 m and its total height, in fact, are functions of time. By discovering the connection between the above named dimensions of tree and time, an impressive method for growth and yield estimation can be obtained. Mathematical models such as this one of tree trunk form may be used like basic units for designing very complicated models. The basic units can be supplemented by the distribution of trees by their dimensions as a partly random function of time. The connection between thickness and height can also be added to the basic units. Designing a mathematical model of a forest is a great problem for the 21st century.

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