Uncertainty Estimation of the Self-Thinning Process by Maximum-Entropy Principle

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Abstract.—When available information is scarce, the Maximum-Entropy Principle can estimate the distributions of parameters. In our case study, we estimated the distributions of the parameters of the forest self-thinning process based on literature information, and we derived the conditional distribution functions and estimated the 95 percent confidence interval (CI) of the self-thinning process for several tree species. The 95 percent CI indicated that the slope parameter of the so-called self-thinning law can be considered a random variable with a mean value of -3/2.

In ecology, the equation to describe the relationship between plant density and individuals' average biomass is considered the "self-thinning law" (or "self-thinning rule"). Its form is:

$$B = a \cdot D^{b} \quad \text{or} \quad \log(B) = A + b \cdot \log(D) \qquad (1)$$

where B is the average biomass of individual biobody, D is density. A [equaling to log(a)] and b are parameters. This relationship is called a "law" or "rule," not only because its parameter b has a quite stable value, but also because it can be derived from and interpreted by ecological theory (Drew and Flewelling 1977, Hozumi 1980, Zeide 1987). It is also widely applied in forest studies and management (Smith and Hann 1985; Valentine 1985, 1988; Weller 1987).

Many studies showed that the value of parameter b is close to -3/2 (White and Harper 1970, Hutchings 1979, Dean and Long 1985). However, some studies found that the estimated parameter b is not always around -3/2. It behaves like a random variable instead of a constant. Tolerance of a species, age, and site quality can vary the value of parameter b (Zeide 1985, 1987; Weller 1987; McFadden and Oliver 1988; Lonsdale 1990). The other factor that causes this parameter to change is climate, because tolerance property and site quality are both associated with weather conditions (Zeide 1985, 1987). The influence of those factors is usually not considered in a study. This can be considered to be interior uncertainty in studies of this law in specific forest ecosystems. On the other hand, sampling, measuring, grouping, and processing can cause uncertainty in data (Gertner 1991). This is exterior uncertainty. In studies of self-thinning law, the exterior uncertainty may have the same importance as the interior uncertainty, since sampling cannot easily guarantee that all the observations of a collected sample meet the closure assumption, which is the basis of the self-thinning law. Therefore, the estimated parameter b is impacted by both the interior and exterior uncertainty. Thus, it is necessary to be aware of uncertainty in studies of the relationship between average size and density since we cannot eliminate uncertainty by current techniques.

In studies of the self-thinning law, uncertainty will mainly exist in the distribution of average biomass. Whenever density is given, differences in species, climate patterns, and site quality will produce different total and average biomass. In data collection, measuring error will mainly occur in measurement of biomass, since it is much harder to measure than density. In Eq. 1, density is the independent variable and average biomass is the dependent variable. The estimated parameters will influence only biomass and not density. Therefore, uncertainty will be reflected by biomass. To estimate uncertainty of the self-thinning law is to estimate the behavior of biomass. The behavior of biomass is represented by the distribution of biomass at each density. It is almost impossible to directly estimate the distribution of biomass at all the possible densities, because density is actually a continuous variable. The way to estimate uncertainty in the self-thinning law is to estimate the uncertainty of parameters A and b first, and then estimate biomass.

Parameters A and b have been estimated in many studies. Those point estimates have been integrated to draw the histograms of A and b (Weller 1987). Since we know that species can cause variation in parameters A and b, it is not proper to estimate their distribution and analyze uncertainty without considering species. After grouping based on species, the numbers of the estimated parameters A and b are too small to estimate their distributions by traditional statistical estimation methods. Maximum-Entropy Principle (MEP) can estimate distribution of random variables based on insufficient information (Kapur 1989, Woodbury 1993). Applying MEP to

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estimate the distribution of parameters A and b can be one way to analyze the uncertainty of the self-thinning law.

The purpose of this study is to demonstrate the usage of MEP in forestry study by applying it to estimate the distribution of parameters A and b with literature information and to estimate the distribution of biomass to indicate uncertainty in the self-thinning process.

ESTIMATION OF PARAMETERS' DISTRIBUTION BY MEP

Weller (1987) studied many cases of parameters A and b for several species. In his study, Weller took B as the biomass on unit area instead of individual average biomass in Eq. 1. With this change, the value of b should be close to -1/2 (Weller 1987). We take Weller's parameter A, b estimates (see table 1) as literature information in our study. With the data set, we computed the coefficient of correlation between parameters A and b. The coefficient of correlation was -0.045. It is very small. So, parameters A and b can be considered independent.

We estimated sample mean and interval for each species with the estimated A's in table 1. The sample means and intervals are listed in table 2. For parameter b, we assume that its mean is -0.5 and its interval is {-0.8, -0.2}. For convenience, b', the absolute value of b, was used in estimation. Following Laplace's principle of insufficiency, uniform distribution was selected as prior distribution of both A and b'. For species *Pinus taeda, Picea abies, Trifolium subterraneum, Erigeron canadensis,* and *Eucalyptus regnans*, the mean and midpoint are equal in each species (see table 2). According to MEP (Kapur 1989, Woodbury 1993), the joint posterior probability distribution function (p.d.f.) of A and b' is a uniform distribution:

$$f(A, b') = \frac{1}{0.6(A_2 - A_1)} \qquad (2)$$

For the other species, the p.d.f. of A and b' is (Kapur 1989, Woodbury 1993):

$$f(A, b') = \frac{-\beta\exp(-\beta A)}{0.6[\exp(-\beta A_1) - \exp(-\beta A_2)]} \qquad (3)$$

where β can be numerically solved as (Woodbury 1993):

$$\beta \approx \frac{3(A_2 - A_1 - 2\mu_A)}{2\mu_A^2}$$

Table 1.—*Estimates of parameters of self-thinning law (from Weller 1987)*

Species and case number	Intercept [log(a)]	Slope (-b)	Species and case number	Intercept [log(a)]	Slope (-b)
Abies sacchalinsis			*Beta vulgaris*		
(1)	1.71	2.786*	(1)	4.79	0.662*
(2)	4.39	0.465	(2)	5.12	0.692*
(3)	4.16	0.649*	(3)	5.09	0.668*
Erigeron canadensis			(4)	5.22	0.649
(1)	4.36	0.621*	(5)	5.30	0.648
(2)	5.70	1.038*	(6)	6.39	1.335
Eucalyptus regnans			(7)	9.93	2.304
(1)	1.39	2.478*	*Picea abies*		
(2)	3.44	1.066*	(1)	3.97	0.433*
Lolium perenne			(2)	3.90	0.422*
(1)	4.80	0.427	*Pinus strobus*		
(2)	4.20	0.245	(1)	3.34	1.116*
(3)	4.33	0.544*	(2)	3.78	0.724*
(4)	4.28	0.503	(3)	3.44	0.954*
(5)	3.74	0.324	*Pinus taeda*		
Trifolium subterraneum			(1)	4.21	0.305*
(1)	4.60	0.473	(2)	3.42	0.670*
(2)	5.17	0.622			

*By Weller's analysis: the slopes are different from -1/2 at 0.05 level.

Table 2.—*Estimated distribution of parameter A for different species by MEP*

Species	Lower bound(A_1)	Upper bound(A_2)	Mean	β
Abies sacchalinsis	0.06	6.04	3.420	-0.098321
Beta vulgaris	3.16	11.59	5.976	0.526256
Erigeron canadensis	2.71	7.35	5.030	—*
Eucalyptus regnans	-0.26	5.09	2.415	—*
Lolium perenne	2.14	6.45	4.280	0.009826
Picea abies	2.25	5.62	3.935	—*
Pinus strobus	1.69	5.43	3.520	0.035833
Pinus taeda	1.77	5.86	3.815	—*
Trifolium subterraneum	2.95	6.82	4.885	—*

*In those species, their intercepts have uniform distribution.

The computed β values for different species is in table 2 and the estimated marginal distribution of parameter A for all the species are in figure 1.

Distribution Transformation

Assuming the distribution of parameters A and b are $f_1(A)$ and $f_2(b)$, respectively, and their joint distribution is $f(A,b)$, the distribution of biomass can be gained by using the following transformation.

Let $y_1 = A - b' \cdot \log(D)$ and $y_2 = b'$

where b'=-b, A and D have the same meaning as in Eq. 1, and y_1 equals the logarithm of biomass.

We can have:

$A = G_1(y_1, y_2) = y_1 + y_2 \cdot \log(D)$ and

$b' = G_2(y_1, y_2) = y_2$

and Jacobian J=1. Therefore, the joint distribution of y_1 and y_2 can be transformed by:

$$W(y_1, y_2) = f[G_1(y_1, y_2), G_2(y_1, y_2)]|J| \quad (4)$$

and the marginal distribution of y_1 is:

$$W_1(y_1) = \int W(y_1, y_2) dy_2 \quad (5)$$

$W_1(y_1)$ is the distribution of the dependent variable, log(B), in Eq. 1, if the joint distribution of y_1 and y_2 is integrable.

PROBABILITY DISTRIBUTION FUNCTION OF LOG(B)

Based on the posterior distribution of A and b', $W_1(y_1)$ can be gained by integration on y_2 for each type of distribution, since both types of joint distribution of A and b' are integrable. The area of the joint distribution of y_1 and y_2 can be divided into two to three different sections (see fig. 1). Integration was based on the boundary of each section under each relationship between log(D) and the intervals of parameters A and b'.

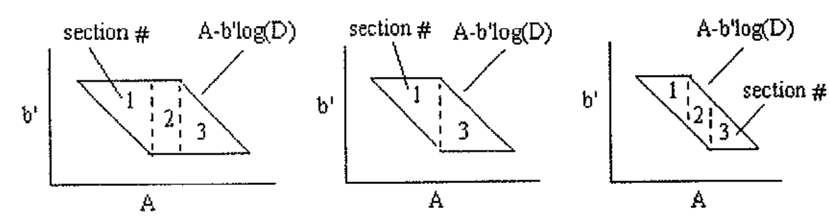

Fig. 1-a Fig. 1-b Fig. 1-c

Figure 1.—*The three integral sections of the joint distribution of* y_1 *and* y_2 *under different relationships between log(D) and the intervals of parameters A and b': Fig. 1-a.* $[0.6\log(D)]^2 < (A_2 - A_1)^2 + 0.6^2$, *Fig. 1-b.* $[0.6\log(D)]^2 = (A_2 - A_1)^2 + 0.6^2$, *and Fig. 1-c.* $[0.6\log(D)]^2 > (A_2 - A_1)^2 + 0.6^2$.

For Uniform Distribution of A and b'

For species *E. canadensis, E. regnans, P. abies, P. taeda,* and *T. subterraneum*, parameters A and b have joint p.d.f. of uniform expressed by Eq. 2. The transformed joint p.d.f. of y_1 and y_2 is:

$$W(y_1,y_2) = f[G_1(y_1,y_2),G_2(y_1,y_2)]|J| = \frac{1}{0.6(A_2 - A_1)} \quad (6)$$

And the marginal distribution of y_1 is :

$$W_1(y_1) = \int \frac{dy_2}{0.6(A_2 - A_1)} \quad (7)$$

When the relationships between A's interval and log(D) is different, the marginal distribution of y_1 will be different within different sections. The integrated generalized $W_1(y_1)$'s are listed in table 3-1.

For Exponential Distribution of A and b'

In species *L. perenne, B. vulgaris, A. sacchalinsis*, and *P. strobus*, parameters A and b' have exponential joint distribution (Eq. 3). The transformed joint p.d.f. of y_1 and y_2 is:

$$W(y_1,y_2) = f[G_1(y_1,y_2),G_2(y_1,y_2)]|J|$$

$$= \frac{-\beta \exp\{-\beta[y_1 + y_2 \log(D)]\}}{0.6[\exp(-\beta A_1) - \exp(-\beta A_2)]} \quad (8)$$

And the marginal p.d.f. of y_1 is :

$$W_1(y_1) = \int W(y_1,y_2)dy_2 \quad (9)$$

Just as with uniform distribution of parameters A and b', integration section determines y_1's marginal p.d.f.. When the intervals of parameter A and log(D) have different relationships, the obtained $W_1(y_1)$'s are listed in table 3-2.

C. D. F. AND CONFIDENCE LEVEL OF LOG(B)

Fortunately, with different relationships between log(D) and intervals of parameters A and b', $W_1(y_1)$ has the same p.d.f. on Section 1 (or Section 3) when the joint distribution of A and b' is the same (see table 3). The difference of p.d.f. occurred in Section 2. Therefore, we can find out the relationship between confidence level and upper and lower bounds on y_1's distribution. Let $2a$ be the confidence level, y_a and y_{1-a} be the lower and upper bounds of $1-2a$ confidence interval, respectively. We will derive the relationships between confidence level and bounds based on the joint distribution of parameters A and b'.

For Uniform Distribution of A and b'

In table 3-1, the $W_1(y_1)$ of Section 1 is in the first row. Its corresponding cummulated probability distribution (C. D. F.) will be:

Table 3-1.—*The probability distribution function of log(B) for those species in which the parameters A and b' have uniform joint distribution*

Relationship	$W_1(y_1)$	Interval
$[0.6\log(D)]^2 < (A_2 - A_1)^2 + 0.6^2$	$W(y_1,y_2)[0.8 - \frac{(A_1 - y_1)}{\log(D)}]$*	$A_1 - 0.8\log(D) < y_1 < A_1 - 0.2\log(D)$
	$0.6 \cdot W(y_1,y_2)$	$A_1 - 0.2\log(D) < y_1 < A_2 - 0.8\log(D)$
	$W(y_1,y_2)[\frac{(A_2 - y_1)}{\log(D)} - 0.2]$	$A_2 - 0.2\log(D) > y_1 > A_2 - 0.8\log(D)$
	0	elsewhere
$[0.6\log(D)]^2 = (A_2 - A_1)^2 + 0.6^2$	$W(y_1,y_2)[0.8 - \frac{(A_1 - y_1)}{\log(D)}]$*	$A_1 - 0.8\log(D) < y_1 < A_1 - 0.2\log(D)$
	$W(y_1,y_2)[\frac{(A_2 - y_1)}{\log(D)} - 0.2]$	$A_2 - 0.2\log(D) > y_1 > A_2 - 0.8\log(D)$
	0	elsewhere
$[0.6\log(D)]^2 > (A_2 - A_1)^2 + 0.6^2$	$W(y_1,y_2)[0.8 - \frac{(A_1 - y_1)}{\log(D)}]$	$A_1 - 0.8\log(D) < y_1 < A_2 - 0.8\log(D)$
	$W(y_1,y_2)\frac{(A_2 - y_1)}{\log(D)}$	$A_2 - 0.8\log(D) < y_1 < A_1 - 0.2\log(D)$
	$W(y_1,y_2)[\frac{(A_2 - y_1)}{\log(D)} - 0.2]$	$A_2 - 0.2\log(D) > y_1 > A_1 - 0.2\log(D)$
	0	elsewhere

*$W(y_1,y_2)$ is the joint distribution of y_1 and y_2.

Table 3-2.—*The probability distribution function of log(B) for those species in which the parameters A and b' have exponential joint distribution.*

Situation	$W_1(y_1)$*	Interval
$[0.6\log(D)]^2 < (A_2 - A_1)^2 + 0.6^2$	$\frac{G_1(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_1 - 0.8\log(D) < y_1 < A_1 - 0.2\log(D)$
	$\frac{G_2(A_1, A_2)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_1 - 0.2\log(D) < y_1 < A_2 - 0.8\log(D)$
	$\frac{G_3(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_2 - 0.2\log(D) > y_1 > A_2 - 0.8\log(D)$
	0	elsewhere
$[0.6\log(D)]^2 = (A_2 - A_1)^2 + 0.6^2$	$\frac{G_1(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_1 - 0.8\log(D) < y_1 < A_1 - 0.2\log(D)$
	$\frac{G_3(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_2 - 0.2\log(D) > y_1 > A_2 - 0.8\log(D)$
	0	elsewhere
$[0.6\log(D)]^2 > (A_2 - A_1)^2 + 0.6^2$	$\frac{G_1(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_1 - 0.8\log(D) < y_1 < A_2 - 0.8\log(D)$
	$\frac{G_4(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_2 - 0.8\log(D) < y_1 < A_1 - 0.2\log(D)$
	$\frac{G_3(A_1, A_2, y_1)\exp(-\beta y_1)}{0.6\ln(D)[\exp(-\beta A_1) - \exp(-\beta A_2)]}$	$A_2 - 0.2\log(D) > y_1 > A_1 - 0.2\log(D)$
	0	elsewhere

*In $W_1(y_1)$, $G_1(A_1, A_2, y_1) = \exp[-0.8\beta\log(D)] - \exp[-\beta(A_1 - y_1)]$,

$G_2(A_1, A_2) = \exp[-0.8\beta\log(D)] - \exp[-0.2\beta\log(D)]$,

$G_3(A_1, A_2, y_1) = \exp[-\beta(A_2 - y_1)] - \exp[-0.2\beta\log(D)]$,

and $G_4(A_1, A_2, y_1) = \exp[-\beta(A_2 - y_1)] - \exp[-\beta(A_1 - y_1)]$.

$$P(Y_1) = \int_{-\infty}^{Y_1} W_1(y_1)dy_1 = \frac{1.667Y_1[A_1 - 0.8\log(D)] - 0.833\{[A_1 - 0.8\log(D)]^2 - Y_1^2\}}{0.8\log(D)(A_1 - A_2)}$$

$$A_1 - 0.8\log(D) < Y_1 < \min\{[A_1 - 0.2\log(D)],[A_2 - 0.8\log(D)]\}$$

After setting confidence level 2α, the lower bound will be:

$$y_\alpha = A_1 - 0.8\log(D) + \sqrt{1.2 \cdot \alpha \cdot \log(D)(A_2 - A_1)}, \quad \log(D)>0 \tag{10}$$

The $W_1(y_1)$ of Section 3 is in the third row. Its corresponding C. D. F. will be:

$$P(Y_1) = 1 - \int_{Y_1}^{\infty} W_1(y_1)dy_1 = 1 - \frac{0.833\{[A_2 - 0.2\log(D)]^2 - Y_1^2\} - 1.667\{Y_1[0.2\log(D) - A_1] - [A_2^2 + 0.04\log^2(D) - 0.4A_2\log(D)]\}}{\log(D)(A_1 - A_2)}$$

$$A_2 - 0.2\log(D) > Y_1 > \max\{[A_1 - 0.2\log(D)],[A_2 - 0.8\log(D)]\}$$

Accordingly, the upper bound will be:

$$y_{1-\alpha} = A_2 - 0.2\log(D) - \sqrt{1.2 \cdot \alpha \cdot \log(D)(A_2 - A_1)}, \quad \log(D)>0 \tag{11}$$

For Exponential Distribution of A and b'

Based on the $W_1(y_1)$ of Section 1, the corresponding C. D. F. will be:

$$P(Y_1) = \int_{-\infty}^{Y_1} W_1(y_1)dy_1$$

$$= \frac{1.33[e^{\beta A_2}(1.25+1.25\beta A_1 - \beta \log(D)]}{\beta \log(D)(e^{\beta A_1} - e^{\beta A_2})} +$$

$$\frac{1.667\{e^{\beta[A_1+A_2-0.8\log(D)]} + Y_1\beta e^{\beta(A_2+Y_1)}\}}{\beta \log(D)e^{\beta Y_1}(e^{\beta A_2} - e^{\beta A_1})},$$

$$A_1 - 0.8\log(D) < Y_1 < \min\{[A_1 - 0.2\log(D)],[A_2 - 0.8\log(D)]\}$$

And the C. D. F. on Section 3 is:

$$P(Y_1) = 1 - \int_{Y_1}^{\infty} W_1(y_1)dy_1 = 1 - \frac{1.667}{\log(D)}\{\frac{e^{\beta A_1}}{(e^{\beta A_1} - e^{\beta A_2})}[\frac{1}{\beta} - A_2$$

$$+0.2\log(D)] - \frac{[e^{\beta[A_1+A_2-0.2\log(D)]} + Y_1\beta e^{\beta(A_1+Y_1)}]}{\beta e^{\beta Y_1}(e^{\beta A_1} - e^{\beta A_2})}\},$$

$$A_2 - 0.2\ln(D) > Y_1 > \max\{[A_1 - 0.2\log(D)],[A_2 - 0.8\log(D)]\}$$

Because the C. D. F.'s at both sides are complicated, it is very difficult to derive an explicit relationship between the confidence level and bounds. However, when a confidence level is given, both the lower and upper bound of the confidence interval can be obtained by numerical methods based on the two C. D. F.'s.

UNCERTAINTY OF THE SELF-THINNING PROCESS

A 95 percent confidence level was chosen to determine the bounds of log(B) at different densities for each species. The confidence area of each species is shown in figure 2. In a species, the self-thinning process described by each case from Weller (1987) was also drawn in figure 2. For the 28 cases from the nine studied species, the proportion of the cases that do not violate self-thinning law is quite high. In the cases from *P. taeda*, *P. abies*, *E. canadensis*, *T. subterraneum*, and *L. perenne*, their estimated self-thinning processes are totally covered by their corresponding 95 percent confidence area (see fig. 2). In Case 7 of *B. vulgaris* and Case 1 of *P. strobus*, self-thinning processes cannot be covered by the corresponding 95 percent confidence area when log(D) is larger than 3. In Case 1 of either *A. sacchalinsis* or *E. regnans*, the self-thinning process is outside the corresponding 95 percent confidence area. Of all the 28 cases, just 4 fail to follow the self-thinning law based on 95 percent confidence area. The proportion not supporting the self-thinning law is 14.3 percent. In 85.6 percent of the cases, the self-thinning law cannot be rejected.

CONCLUSION AND DISCUSSION

Maximum-Entropy Principle (MEP) provides an alternative to estimate the distribution of random variables with insufficient information. Using probability transformation, distribution and uncertainty of biomass can be obtained from the uncertainty of parameters A and b of the self-thinning law based on literature information. With uncertainty in nature, sampling, and parameter estimation method, parameter b of the self-thinning law should be a random variable instead of a constant. The estimated 95 percent CI areas of the studied species suggested that the mean value of b should be –3/2.

LITERATURE CITED

Dean, T.J.; Long, J.N. 1985. Response of self-thinning to artificially reduced levels of leaf area in monocultures of Trifolium pratense. Annals of Botany. 55: 361-365.

Drew, T.J.; Flewelling, J.W. 1977. Some recent theories in yield-density and their application to Monterey pine plantation. Forest Science. 23: 517-534.

Gertner, G.Z. 1987. Approximating precision in simulation projections: an alternative to Monte Carlo methods. Forest Science. 33: 230-239.

Gertner, G.Z. 1991. Prediction bias and response surface curvature. Forest Science. 37: 755-765.

Hozumi, K. 1980. Ecological and mathematical considerations on self-thinning in even- pure stands, 2: growth analysis of self-thinning. Botanical Magazine. 93: 149-166.

Hutchings, M.J. 1979. Weight-density relationships in ramet populations of clonal perennial herbs, with special reference to the -3/2 power law. Journal of Ecology. 67: 21-33.

Jaynes, E.T. 1957. Information theory and statistical mechanics. Physiological Reviews. 106: 620-630.

Lonsdale, W.M. 1990. Self-thinning rule: dead or alive? Ecology. 71: 1373-1388.

McFadden, G.; Oliver, C.D. 1988. Three-dimensional forest growth model relating tree size, tree number, and stand age: relation to previous growth models and to self-thinning. Forest Science. 34: 662-676.

Kapur, J.N. 1989. Maximum-entropy models in science and engineering. New York, NY: John Wiley and Sons.

Kullback, S. 1959. Information theory and statistics. New York, NY: John Wiley and Sons.

Schreuder, H.T.; Gregoire, T.G.; Wood, G.B. 1992. Sampling methods for multiresource forest inventory. New York, NY: John Wiley & Sons.

Smith, N.J.; Hann, D.W. 1985. A new analytical model based on the -3/2 power rule of self-thinning. Canadian Journal of Forest Research. 14: 605-609.

Ulrych, T.; Bassrei, A.; Lane, M. 1990. Minimum relative entropy inversion of 1D data with applications. Geophysical Prospect. 38: 465-487.

Valentine, H.T. 1988. A carbon-balance model of stand growth: a derivation employing pipe model theory and self-thinning rule. Annals of Botany. 62: 389-396.

White, J.; Harper, J.L. Correlated changes in plant size and number in plant populations. Journal of Ecology. 67: 21-33.

Figure 2.—*The estimated 95 percent CI of log(B) of each species and its individual cases. Fig. 2-a. legend of Figure 2, Fig. 2-b. species:* A. sacchalinsis, *Fig. 2-c. species:* B. vulgaris, *Fig. 2-d. species:* E. canadensis; *Fig. 2-e. species:* E. regnans, *Fig. 2-f. species:* L. perenne, *Fig. 2-g. species:* P. abies, *Fig. 2-h. species:* P. strobus, *Fig. 2-i. species:* P. taeda, *Fig. 2-j. species:* T. subterraneum.

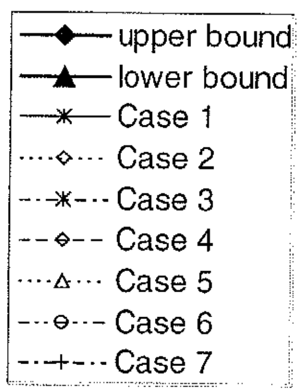

Fig. 2-a.

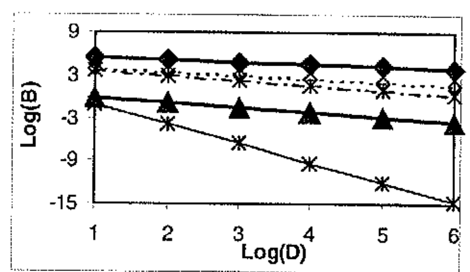

Fig. 2-b

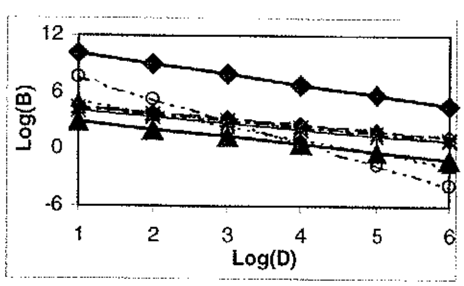

Fig. 2-c

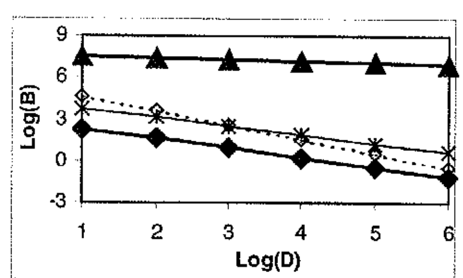

Fig. 2-d

(Figure 2 continued on next page)

(Figure 2 continued)

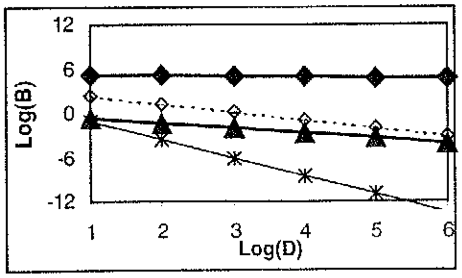

Fig. 2-e

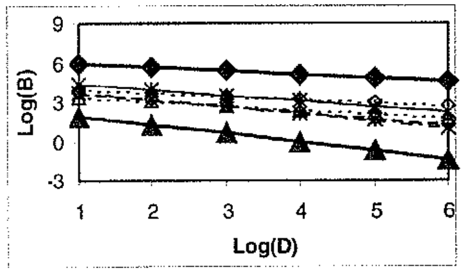

Fig. 2-f

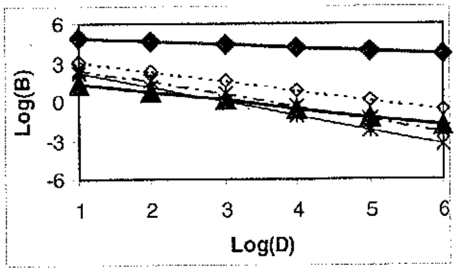

Fig. 2-g

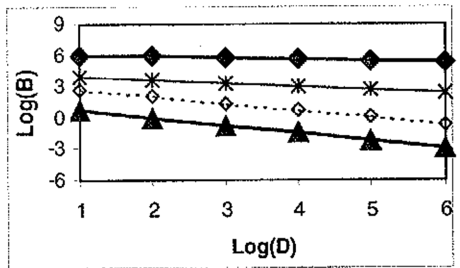

Fig. 2-h

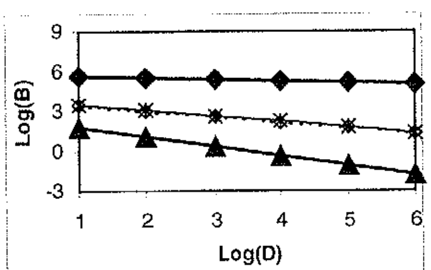

Fig. 2-i

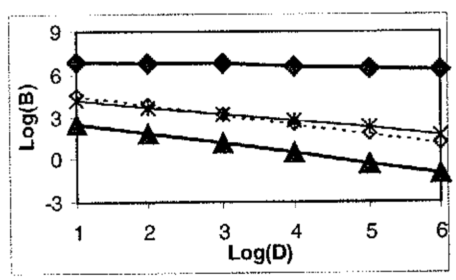

Fig. 2-j

Woodbury, A.D. 1993. Minimum relative entropy: forward probabilistic modeling. Water Resources Research. 29(8): 2847-2860.

Weller, D.E. 1987. A reevaluation of the -3/2 power rule of plant self-thinning. Ecological Monographs. 57: 23-43.

Zeide, B. 1985. Tolerance and self-tolerance of trees. Forest Ecology and Management. 13: 149-166.

Zeide, B. 1987. Analysis of the -3/2 power law of self-thinning. Forest Science. 33: 517-537.